PMM U.S.S.R.,Vol.46,pp.556-558
0021-8928/83/4 0556 $7.50/0

UDC 62-50

OPTIMIZATION OF A QUADRATIC PERFORMANCE INDEX ON SOLUTIONS OF NONLINEAR TWO-POINT BOUNDARY-VALUE PROBLEMS*

V.V. ANISOVICH

A problem of optimizing nonlinear systems with boundary conditions of general form and a quadratic performance index is examined. Conditions for the existence of an optimal control are formulated. The optimal controls are written in explicit form as functions of the phase coordinate and the solutions of auxiliary boundary-value problems. Various applied problems of mechanics lead to the determination of optimal modes of the systems being examined /1/.

1. We consider a controlled system of differential equations with boundary conditions

$$x^{\cdot}(t) = A(t)x(t) + B(x, t)u(t),\ t \in [t_0, t_1] \qquad (1.1)$$
$$g(x(t_0),\ x(t_1)) = 0$$

Here $x(t)$ is an n-dimensional vector-valued function, $u(t)$ is an m-dimensional piecewise-continuous vector-valued function, $A(t)$ and $B(x, t)$ are continuous matrices of dimensions $n \times n$ and $n \times m$, respectively, and $g(y, z)$ is an n-dimensional continuous vector-valued function. We remark that sometimes the boundary condition in (1.1) consists of two relations at the beginning $t = t_0$ and the end $t = t_1$ of the control process, of dimensions m_0 and m_1, respectively,

$$g_0(x(t_0)) = 0,\ g_1(x(t_1)) = 0,\ m_2 + m_1 = n$$

In particular, if we formally admit that one of the quantities m_0 or m_1 equals zero, then Eqs. (1.1) have the form of a Cauchy problem.

From the whole set of piecewise-continuous $u(t) \in R^m$ we examine those $u(t) \in U \subseteq R^m$ to which correspond at least one solution $x(t)$ of boundary-value problem (1.1) and a functional (the asterisk denotes transposition)

$$J(x, u) = \int_{t_0}^{t_1} (u^*Ku + u^*L^*x + x^*Lu + x^*Mx^*)\,dt \qquad (1.2)$$

taking a finite value. We assume that U is a nonempty set. Here $K(t)$, $L(t)$, $M(t)$ are continuous matrices of appropriate dimensions, while $K(t)$ is positive definite, and $K(t)$, $M(t)$ are symmetric. We are required to find a control $\bar{u}(t) \in U$ and the corresponding solution $\bar{x}(t)$ of boundary-value problem (1.1) such that functional (1.2) takes the least value. The control $\bar{u}(t)$ thus found is called optimal.

In applications it is important /2/ to find an optimal control $\bar{u} = \bar{u}(\bar{x}, t)$ as a function of its corresponding solution $\bar{x}(t)$, i.e., to solve the optimal control synthesis problem. The application of dynamic programming for synthesis problems lead to nonstandard equations /1,2/ the solving of which gives rise to known difficulties. To find the synthesizing optimal control $\bar{u} = \bar{u}(\bar{x}, t)$ of problem (1.1), (1.2) we use the approach in /3/. We introduce an auxiliary matrix $N(t)$ of dimension $n \times n$ and, with due regard to (1.1), we reduce functional (1.2) to the canonic form. We consider the system of boundary-value problems

$$N^{\cdot} = NBK^{-1}B^*N + N(BK^{-1}L^* - A) + (LK^{-1}B^* - A^*)N + LK^{-1}L^* - M \qquad (1.3)$$
$$x^*(t_0)N(t_0)x(t_0) - x^*(t_1)N(t_1)x(t_1) = 0,\quad B = B(x, t)$$
$$x^{\cdot} = Ax - BK^{-1}B^*Nx - BK^{-1}L^*x,\ g(x(t_0),\ x(t_1)) = 0$$

and assume that (1.3) has at least one solution $(\bar{x}(t),\ \bar{N}(t))$.

Theorem. When all the above-mentioned conditions are fulfilled the synthesizing optimal control for problem (1.1), (1.2) exists and is computed by the formula

$$\bar{u}(\bar{x}, t) = -K^{-1}[B^*(\bar{x}, t)\bar{N} + L^*]\bar{x} \qquad (1.4)$$

*Prikl.Matem.Mekhan.,46,No.4,pp.697-699,1982

where $\inf J(x, u) = J(\bar{x}, \bar{u}) = 0$.

Proof. The matrix $N(t)$ is symmetric because of the symmetry of the first boundary-value problem in system (1.3). Indeed, by transposing the first boundary-value problem in (1.3) and subtracting, we obtain

$$d(N - N^*)/dt = (N - N^*) BK^{-1}B^*N + N^*BK^{-1}B^*(N - N^*) + (N - N^*)(BK^{-1}L^* - A) + (LK^{-1}B^* - A^*)(N - N^*)$$
$$x^*(t_0)[N(t_0) - N^*(t_0)]x(t_0) = x^*(t_1)[N(t_1) - N^*(t_1)]x^*(t_1) \tag{1.5}$$

Problem (1.5) has the trivial solution $N(t) - N^*(t) = 0$; consequently, $N(t) = N^*(t)$. Just as in /4/, expressing M from the first equation of system (1.3) and then Ax from (1.1) and allowing for the symmetry of matrices K, M, N, we obtain

$$x^* Mx = x^*NBK^{-1}B^*Nx + x^* NBK^{-1}L^*x + x^*NBu + x^*LK^{-1}B^*Nx + u^*B^*Nx + x^*LK^{-1}L^*x - d(x^*Nx)/dt \tag{1.6}$$

Using (1.6), we write functional (1.2) in canonical form

$$J(x, u) = \int_{t_0}^{t_1} G^*KG\,dt, \quad G = u + K^{-1}L^*x + K^{-1}B^*(x, t)Nx \tag{1.7}$$

Because matrix K is positive definite we have $J(x, u) \geqslant 0$; consequently, the smallest value of functional (1.7) is reached on the vector $G = 0$. Thus the optimal control is computed by formula (1.4). Substituting (1.4) into (1.1), we obtain the second boundary-value problem in system (1.3), which yields the possibility of determining $(\bar{x}(t), \bar{N}(t))$ from (1.3). The theorem is proved.

Notes. 1°. The requirement of the existence of the solution of boundary-value problem (1.3) restricts the class being examined of problems (1.1), (1.2) since in the general case the minimal value of functional (1.2) can be both positive (when $L = 0, M \geqslant 0$) as well as negative (when the norm $\|L\|$ is sufficiently large). However, although the solution obtained does not exhaust all possible cases, the representation of the control in the form (1.4) has well-known advantages /1,2/. The application of the necessary optimality conditions in /1,2/ together with the sufficient condition obtained permits us to lessen the number of solutions looked at for optimality. The question of the existence of at least one solution of (1.3) is resolved with the aid of well-known criteria /5/. When a small parameter is present in the right-hand side of Eq.(1.1) we can apply asymptotic methods /6/.

2°. The computation of optimal control (1.4) reduces to solving the system of boundary-value problems in (1.3), which in applications are solved numerically.

3°. In the presence of several optimal controls (1.4) and of the corresponding solutions of problem (1.1) furnishing a zero value to (1.2) there arise the questions of choosing a practically-realizable stable mode, of delineating the capture domain for each of the modes, etc. In this case additional investigations are made, using the physical properties of the controlled object.

4°. As was noted in /7/, the question of designing systems having optimal periodic, almost-periodic, and, in particular, quasiperiodic motions has scarcely been studied. Systems of such type describe a number of important applied problems of mechanics, chemical engineering, cardiology, etc. (see /7/ and the bibliography presented therein). Obviously /6/, the systems indicated can be written in form (1.1). For the verification of condition (1.3), for example, in the almost-periodic case there exist a number of criteria /8/. When a small parameter is present in the right-hand side of Eqs.(1.3) we can use the existence theorems derived in /6/.

2. Let a controlled process with performance index (1.2) be described by the boundary-value problem

$$x^\circ(t) = A(t)x(t) + u(t) + \varphi(x, t), \quad t \in [t_0, t_1], \quad g(x(t_0) \quad x(t_1)) = 0 \tag{2.1}$$

Here $\varphi(x, t)$ is a continuous n-dimensional vector-valued function. We seek a control $\bar{u}(t) \in U$ such that functional (1.2) takes the smallest value on the solutions $\bar{x}(t)$ of problem (2.1). We replace the problem (1.2), (2.1) being examined by the following equivalent optimization problem:

$$x^\circ(t) = A(t)x(t) + B(x, t)v(t), \quad t \in [t_0, t_1], \quad g(x(t_0), \quad x(t_1)) = 0 \tag{2.2}$$

$$I(x, v) = \int_{t_0}^{t_1} (v^*Kv + x^*Lv + v^*Lx + x^*Mx)\,dt \to \inf$$

$$B(x(t), t)\, v(t) = u(t) + \varphi(x(t), t)$$

The scalar function $B = B(x, t)$ is determined on the basis of the equality

$$u^*Ku + x^*Lu + u^*Lx = v^*Kv + x^*L^*v + v^*Lx \qquad (2.3)$$

Relation (2.3) ensures the equality of the integrands in (1.2) and (2.2) at the expense of choosing $B = B(x, t)$; consequently $\inf J(x, u) = \inf I(x, v)$, i.e., the introduction of the n-dimensional vector-valued function $v(t)$ and of the scalar function B by means of the last equality in (2.2) and of (2.3) enables us to write problem (1.2), (2.1) in the form of (1.1), (1.2). The optimal control of problem (2.2) is computed, under fulfillment of the theorem's conditions, by formula (1.4).

From the last equality in (2.2) we find

$$u(x, t) = -B(x, t) K^{-1} [B(x, t) N + L^*] x - \varphi(x, t) \qquad (2.4)$$

Using (2.3) we obtain the equation in B. Let $B = \Phi(x, N, t)$ be the solution of (2.3). Then, substituting $B = \Phi(x, N, t)$ into system (1.3), we determine $(\bar{x}(t), \bar{N}(t))$, with the aid of which we find the optimal control of problem (1.2), (2.1)

$$\bar{u}(\bar{x}, t) = -\Phi(\bar{x}, \bar{N}, t) K^{-1} [\Phi(\bar{x}, \bar{N}, t) N + L^*] \bar{x} - \varphi(\bar{x}, t)$$

In engineering problems, for instance, in the problem of parts orientation by a variable magnetic field /9/, in servomechanisms with backlash, /10/, etc., the "external" generalized forces are frictional forces, for example, viscous /9,10/. The dynamics of such systems are described by a dimensionless first-order scalar equation $x^a + x = uB(x)$, where $x(t)$ is the generalized velocity, $u(t)$ is the controlled action, $B(x(t))$ is the characteristic of the external forces. If the external force is $B(x) = \sqrt{x}$, then the problem of optimizing a system with periodic velocity and with a power performance index can be formulated as

$$x^\circ = -x + u\sqrt{x}, \quad x(0) = x(T), \quad J(x, u) = \int_0^T (u^2 - x^2)\,dt \to \inf$$

Using the theorem in Sect.1, we obtain the optimal solution $\bar{x}(t) = 1$, $\bar{u}(t) = 1$ (the trivial rest case $x(t) = u(t) = 0$ is not examined).

REFERENCES

1. BRYSON A.E. and HO Y.-C., Applied Optimal Control. Waltham, MA, Blaisdell Publ. Co., 1969.
2. MOISEEV N.N., Elements of the Theory of Optimal Systems. Moscow, NAUKA, 1975.
3. ANISOVICH V.V., KRIUKOV B.I. and MADORSKII V.M., On one approach to solving optimal control problems. Dokl. Akad. Nauk SSSR, Vol.251, No.2, 1980.
4. ANISOVICH V.V., Optimal control synthesis in one minimax problem of the optimization of periodic processes. Avtom. i Telemekh., No.11, 1979.
5. LIONS J.-L., Some Methods for Solving Nonlinear Boundary-Value Problems. Moscow, MIR, 1972.
6. AKULENKO L.D., Asymptotic solution of two-point boundary-value problems. PMM Vol.44, No.4, 1980.
7. TONKOV E.L., Optimal periodic motions of a controlled system. In: Mathematical Physics, No.21, Kiev, NAUKOVA DUMKA, 1977.
8. KRASNOSEL'SKII M.A., BURD V.Sh. and KOLESOV Iu.S., Nonlinear Almost-Periodic Oscillations Moscow, NAUKA, 1970.
9. VETIUKOV M.M. and KHODZHAEV K.Sh., Equations of slow motions of systems with quasicyclic coordinates and of electromechanical systems. In: Dynamics of Systems, No.9, Gor'kii, Izd. Gor'kovsk. Univ., 1976.
10. BAUTIN N.N. and LEONTOVICH E.A., Methods and Means of Qualitative Investigation of the Dynamics of Systems on a Plane. Moscow, NAUKA, 1976.

Translated by N.H.C.